On Multiplier Hermitian Structures on Compact Kähler Manifolds

Qi Li

Department of Mathematics
Columbia University, New York, NY 10027

1 Introduction

In [9], Mabuchi introduced the notion of a multiplier Hermitian structure on Kähler manifolds and a generalization of the notions of Kähler-Einstein metric and Kähler-Ricci soliton. In this note we study these new notions on compact Kähler manifolds whose first Chern class is positive.

Let M be a compact Kähler manifold of complex dimension n with positive first Chern class $c_1(M) > 0$ and let $\mathcal{K}$ denote the set of all Kähler forms ω on M in the class $c_1(M)$. Assume that X is a holomorphic vector field on M and that

$$\mathcal{K}_X := \{\omega \in \mathcal{K} : L_{X_{\mathbf{R}}}\omega = 0\} \neq \emptyset$$

where $X_{\mathbf{R}} = X + \overline{X}$ denotes the real vector field on M associated to X. In this note, all Kähler metrics considered will be in this set. We assume also that X is Hamiltonian, i.e., that we can find a function $u_\omega \in C^\infty(M)_{\mathbf{R}}$ normalized by $\int_M u_\omega \omega^n = 0$ so that

$$X^\alpha = \frac{1}{\sqrt{-1}} g^{\alpha\overline{\beta}} \partial_{\bar{\beta}} u_\omega \tag{1.1}$$

where $\omega = \sqrt{-1}\sum_{\alpha,\beta} g_{\overline{\beta}\alpha} dz^\alpha \wedge d\bar{z}^\beta$. In [5], Futaki and Mabuchi proved that

$$l_0 := \min_M u_\omega, \ l_1 := \max_M u_\omega$$

are independent of the choice of $\omega \in \mathcal{K}_X$.

Let σ be a real-valued smooth function defined on interval $[l_0, l_1]$ satisfying one of the following conditions:

(a) $\dot{\sigma} \leq 0 \leq \ddot{\sigma}$
(b) $\ddot{\sigma} > 0$

Here $\dot{\sigma}$ and $\ddot{\sigma}$ are the first derivative and second derivative of σ. Associated to this σ and X, Mabuchi introduced the following generalization of the notions of Kähler-Einstein metric and Kähler-Ricci soliton,

Definition 1. *Let M be a Kähler manifold with $c_1(M) > 0$ and a holomorphic vector field X. Fix a real-valued function σ as above. A metric ω in the class $c_1(M)$ is said to be an Einstein-Mabuchi metric of type X and σ if*

$$Ric(\omega) + \sqrt{-1}\partial\bar{\partial}\sigma(u_\omega) = \omega. \tag{1.2}$$

Remark: In the definition, $Ric(\omega) + \sqrt{-1}\partial\bar{\partial}\sigma(u_\omega)$ can be viewed as the Ricci curvature of the metric $\exp(-\frac{\sigma(u_\omega)}{n})\omega$ which is the multiplier Hermitian metric introduced in [8].

Special cases of Einstein-Mabuchi metrics include:

(1) Kähler-Einstein metrics, corresponding to $\sigma = 0$;

(2) Kähler-Ricci soliton, defined by $Ric(\omega) - \omega = L_X\omega$, where L_X is the Lie derivative along X. This corresponds to the Einstein-Mabuchi metric of type $\sigma(s) = -s + C$

(3) Let h_ω be the Ricci potential, defined by $Ric(\omega) - \omega = \sqrt{-1}\partial\bar{\partial}h_\omega$. If $1 - e^{h_\omega}$ defines a holomorphic vector field as in (1.1), then the metric ω is called the generalized Kähler-Einstein metric with nonvanishing Futaki invariant [6]. This metric corresponds to the Einstein-Mabuchi metric of type $\sigma(s) = -\log(s + C)$ where C is a constant strictly greater than l_0.

According to a well-known conjecture of Yau [15], the existence of Kähler-Einstein metrics should be equivalent to a notion of stability in geometric invariant theory. Formulations of versions of this conjecture in terms of the notion of K-stability have been given by Tian [11] and Donaldson [4]. Analytically, the existence of Kähler-Einstein metrics is related to the properness of the functional F_ω. By properness we mean that for any sequence $\phi_i \in C^\infty(M)_{\mathbf{R}}$ such that $\omega_{\phi_i} = \omega + \sqrt{-1}\partial\overline{\partial}\phi_i > 0$, we must have $\limsup_{i\to\infty} F_\omega(\phi_i) = +\infty$ whenever $\lim_{i\to\infty} J_\omega(\phi_i) = +\infty$. Here

$$\begin{aligned} J_\omega(\phi) &:= \frac{1}{V}\int_0^1\int_M \dot{\phi}_s(\omega^n - \omega^n_{\phi_s})ds \\ F_\omega(\phi) &:= J_\omega(\phi) - \frac{1}{V}\int_M \phi\omega^n - \log\left(\frac{1}{V}\int_M e^{h_\omega - \phi}\omega^n\right) \end{aligned}$$

for all function ϕ such that $\omega_\phi > 0$. In the definition of $J_\omega(\phi)$, ϕ_s is a path connecting 0 and ϕ with $\phi_0 = 0$ and $\phi_1 = \phi$. We shall also require the functional $I_\omega(\phi)$, which is closely related to $J_\omega(\phi)$ and is defined by

$$I_\omega(\phi) := \frac{1}{V}\int_M \phi(\omega^n - \omega^n_\phi).$$

In [11], Tian proved that there exists a Kähler-Einstein metric ω_{KE} on the Kähler manifold (M, ω) with $c_1(M) > 0$ as long as the functional F_ω is proper. More precisely, he proved an inequality of Moser-Trudinger type for Kähler-Einstein manifolds M without nontrivial holomorphic vector fields, i.e.,

$$F_{\omega_{KE}}(\phi) \geq AJ_{\omega_{KE}}(\phi)^\gamma - B \tag{1.3}$$

where $\gamma = \frac{e^{-n}}{8n+8+e^{-n}}$. Clearly, this inequality implies that the functional $F_{\omega_{KE}}(\phi)$ is proper. Recently it was proved in [10] by Phong-Song-Sturm-Weinkove that the exponent γ can be taken to be 1. In [3], the results of [11] have been extended, under some additional assumptions, to the case of Kähler-Ricci solitons by Cao-Tian-Zhu with $\gamma = 1/4n + 5$ for the generalized functionals $\tilde{I}$, $\tilde{J}$, $\tilde{F}$ associated to the vector field X.

In this note we generalize the results in [11], [3], [10] and [13] to Einstein-Mabuchi metrics. First, we introduce the appropriate generalizations of the functionals I, J and F, which we still denote by $\tilde{I}_\omega(\phi), \tilde{J}_\omega(\phi), \tilde{F}_\omega(\phi)$. Note that $\tilde{I}_\omega(\phi)$ and $\tilde{J}_\omega(\phi)$ first appeared in [9]. A key feature of these generalizations is the use of the volume form $e^{-\sigma(u_\omega)}\omega^n$ instead of the volume form ω^n. Without loss of generality, we may assume that $V = \int_M e^{-\sigma(u_\omega)}\omega^n = \int_M \omega^n$. We set

$$\tilde{I}_\omega(\phi) := \frac{1}{V}\int_M \phi(e^{-\sigma(u_\omega)}\omega^n - e^{-\sigma(u_{\omega_\phi})}\omega_\phi^n) \tag{1.4}$$

$$\tilde{J}_\omega(\phi) := \frac{1}{V}\int_0^1\int_M \dot{\phi}_s(e^{-\sigma(u_\omega)}\omega^n - e^{-\sigma(u_{\omega_{\phi_s}})}\omega_{\phi_s}^n)ds \tag{1.5}$$

$$\tilde{F}_\omega(\phi) := \tilde{J}_\omega(\phi) - \frac{1}{V}\int_M \phi e^{-\sigma(u_\omega)}\omega^n - \log(\frac{1}{V}\int_M e^{h_\omega-\phi}\omega^n) \tag{1.6}$$

The variational derivative of the functional $\tilde{F}_\omega(\phi)$ is readily computed

$$\delta\tilde{F} = \int_M \delta\phi(e^{-\sigma(u_{\omega_\phi})}\omega_\phi^n - \frac{1}{\int_M e^{h_\omega-\phi}\omega^n}e^{h_\omega-\phi}\omega^n) \tag{1.7}$$

Thus the critical points ϕ of the functional $\tilde{F}_\omega(\phi)$ are given by the equation

$$e^{-\sigma(u_{\omega_\phi})}\omega_\phi^n - \frac{V}{\int_M e^{h_\omega-\phi}\omega^n}e^{h_\omega-\phi}\omega^n = 0 \tag{1.8}$$

This is the equation for Einstein-Mabuchi metrics as we shall see in the next section.

Let $Aut^0(M)$ be the identity component of the group of all holomorphic automorphisms of M, and let $G \subset Aut^0(M)$ be a maximal compact subgroup. Let $Z(X)$ be the compact subgroup of G consisting of all $g \in G$ such that $Ad(g)X = X$, and let $Z^0(X)$ be the identity component of $Z(X)$. Let $\mathcal{H}_X$ be also the set of all $X_{\mathbf{R}}$ invariant functions ϕ in $C^\infty(M)_{\mathbf{R}}$ such that ω_ϕ is in $\mathcal{K}_X$. In [9], Mabuchi proved that Einstein-Mabuchi metrics on M with respect to X must be $Z^0(X)$-invariant. We introduce the following definition of properness.

Definition 2 *The functional $\tilde{F}_\omega$ is said to be proper with respect to the functional $\tilde{J}_\omega$ if for any sequence $\{\phi_i\}$ of $Z^0(X)$-invariant functions with $\omega_{\phi_i} \in \mathcal{K}_X$, we have* $\limsup_{i\to\infty}\tilde{F}_\omega(\phi_i) = +\infty$ *whenever* $\lim_{i\to\infty}\tilde{J}_\omega(\phi_i) = +\infty$.

In this note, we will establish the following theorems.

Theorem 1 *If the functional $\tilde{F}_\omega(\phi)$ is proper with respect to the functional $\tilde{J}_\omega(\phi)$, then there exists an Einstein-Mabuchi metric on the Kähler manifold (M,ω).*

Theorem 2 *Let M be a compact Kähler manifold with holomorphic vector field X which admits a Einstein-Mabuchi metric ω_{EM} of type σ. Assume that $K \subseteq Z^0(X)$ is a closed subgroup whose centralizer in G is finite, then there are two positive constants A and B such that for any K-invariant function ϕ in $\mathcal{H}_X$,*

$$\tilde{F}_{\omega_{EM}}(\phi) \geq A\tilde{J}_{\omega_{EM}}(\phi) - B \tag{1.9}$$

Remark: The condition that $K \subseteq Z^0(X)$ is a closed subgroup whose centralizer in G is finite is a natural generalization of a condition introduced in [10] for the case of Kähler-Einstein manifolds with nontrivial holomorphic vector fields.

The organization of the note is as follows. In section 2 we review some basic properties of multiplier Hermitian structures and prove Theorem 1. In section 3 we prove Theorem 2 following the method of [11]. In details, we follow closely the exposition of [10]. In the last section we construct a holomorphic invariant of Futaki type which can be viewed as an obstruction to the existence of the Einstein-Mabuchi metric.

2 Proof of Theorem 1

To an arbitrary smooth path $\varphi = \{\phi_t; 0 \leq t \leq 1\}$ in $\mathcal{H}_X$, it corresponds to a one-parameter family of Kähler forms $\omega(t)$ in $\mathcal{K}_X$ by

$$\omega(t) := \omega_{\phi_t} = \omega + \sqrt{-1}\partial\bar{\partial}\phi_t, \quad 0 \leq t \leq 1 \tag{2.1}$$

for $\omega \in \mathcal{K}_X$.

In [9], Mabuchi used the method of continuity to deform a given metric along a path to find the Einstein-Mabuchi metric, i.e.,

$$Ric(\omega_{\phi_t}) + \sqrt{-1}\partial\bar{\partial}\sigma(u_{\omega_{\phi_t}}) = (1-t)\omega + t\omega_{\phi_t} \tag{2.2}$$

which is equivalent to

$$-\sqrt{-1}\partial\bar{\partial}\log(\omega^n_{\phi_t}) + \sqrt{-1}\partial\bar{\partial}\sigma(u_{\omega_{\phi_t}}) = \sqrt{-1}\partial\bar{\partial}t\phi_t - \sqrt{-1}\partial\bar{\partial}\log(\omega^n) - \sqrt{-1}\partial\bar{\partial}h_\omega \tag{2.3}$$

. Then we get the following complex Monge-Ampère equation ,

$$\frac{\omega^n_{\phi_t}}{\omega^n} = e^{h_\omega - t\phi + \sigma(u_{\omega_{\phi_t}})} \tag{2.4}$$

We want to solve the equation at t=1 which gives the Einstein-Mabuchi metric. Since the right hand side of (2.4) is uniformly bounded when $t = 0$, one can solve the equation at $t = 0$ by standard argument for complex Monge-Ampère equation. The implicit function theorem implies that the set $T = \{t : \text{there is a solution at t}, 0 \leq t \leq 1\}$ is open. If one can get a uniform C^0 estimate for ϕ_t for $t \in T$, by the standard argument for complex Monge-Ampère equation one can show the closeness of the set. As in [1], the C^0 estimate is closely related to the functional $\tilde{I}$, $\tilde{J}$, $\tilde{F}$. In [9], Mabuchi has proved the following results,

Theorem [M1]: *For $t \in [1/2, 1]$, we have positive real constants C_0, C_1, independent of the choice of the pair (ω_{ϕ_t}, t) such that*

$$osc(\phi_t) \leq C_0(\tilde{I}_\omega - \tilde{J}_\omega)(\phi_t) + C_1 \tag{2.5}$$

for all (ω_{ϕ_t}, t) with $t \in [1/2, 1]$. Here $osc(\phi_t) = \max(\phi_t) - \min(\phi_t)$.

So to prove Theorem 1, we only need to prove that the properness of the functional $\tilde{F}$ implies that one can derive upper bound for functional $\tilde{I} - \tilde{J}$. Before proving the theorem, we review some basic properties of the multiplier Hermitian structure.

Since we consider the volume form $e^{-\sigma(u_\omega)}\omega^n$ in the functional, we need the following formula for integration by parts,

$$-\int_M (\bar{\partial} u, \bar{\partial} v)_\omega e^{-\sigma(u_\omega)}\omega^n = \int_M (u\overline{(\Delta_\omega + \sqrt{-1}\dot{\sigma}(u_\omega)\bar{X})v})e^{-\sigma(u_\omega)}\omega^n \tag{2.6}$$

for any complex-valued smooth function u, v on M, where $\Delta_\omega = \sum_{\alpha,\beta} g^{\alpha\bar{\beta}}\partial_\alpha\bar{\partial}_\beta$ is the Lapalacian operator for ω. For simplicity we will use the operator $\Box_\omega$ to denote $\Delta_\omega + \sqrt{-1}\dot{\sigma}(u_\omega)\bar{X}$.

One has $u_{\omega_t} = u_\omega + \sqrt{-1}X(\phi_t)$. By using the fact that ϕ_t is $X_{\mathbf{R}}$ invariant, we have

$$u_{\omega_t} = u_\omega - \sqrt{-1}\bar{X}\phi_t \tag{2.7}$$

Let $\dot{\phi}_t$ denote the partial derivative of $\phi(t)$ with respect to t, by using the (2.7), it is easy to verify that,

$$\frac{\partial}{\partial t}(e^{-\sigma(u_\omega)}\omega^n) = (\Box_{\omega_t}\dot{\phi}_t)e^{-\sigma(u_\omega)}\omega^n \tag{2.8}$$

$$\int_M e^{-\sigma(u_{\omega_{\phi_t}})}\omega_t^n = V = \int_M e^{-\sigma(u_\omega)}\omega^n \quad \textit{for all } \omega \in \mathcal{H}_X \tag{2.9}$$

Mabuchi also proved the following properties of the generalized I, J functional:

Fact 1: $0 \leq \tilde{I}_\omega(\phi_t) \leq (m+2)(\tilde{I}_\omega - \tilde{J}_\omega)(\phi_t)) \leq (m+1)\tilde{I}_\omega(\phi_t)$, *where m is a constant depending only on σ.*
Fact 2: *Along the equation, one has*

$$\frac{d}{dt}(\tilde{I}_{\omega_0} - \tilde{J}_{\omega_0})(\phi_t)) = -\int_M (\phi_t \Box_{\omega_{\phi_t}}\dot{\phi}_t)e^{-\sigma(u_{\omega_{\phi_t}})}\omega_t^n = \int_M \{\dot{\phi}_t + \Box_{\omega_{\phi_t}}\dot{\phi}_t\}(\Box_{\omega_{\phi_t}}\dot{\phi}_t)e^{-\sigma(u_{\omega_{\phi_t}})}\omega_t^n \geq 0 \tag{2.10}$$

i.e., $(\tilde{I}_\omega - \tilde{J}_\omega)(\phi_t)$ *is increasing along the equation.*

We refer interested readers to [9] for details of the above properties. We also need the following properties of the functional $\tilde{F}_\omega$ which establishes the relations between those functionals.

Proposition 1 a).$\tilde{F}_\omega(\phi)$ *satisfies the cocyle condition:* $\tilde{F}_\omega(\phi) + \tilde{F}_{\omega_\phi}(\varphi) = \tilde{F}_\omega(\phi+\varphi)$
b).$\tilde{F}_\omega(\phi_t) = -\frac{1}{t}\int_0^t (\tilde{I}_\omega - \tilde{J}_\omega)(\phi_s)ds - \log\big(\frac{1}{V}\int_M e^{h_\omega - \phi_t}\omega^n\big)$

Proof of Proposition 1 a). The proof is similar to the classical case.
b). By the definition of $\tilde{F}_\omega$, it suffices to prove that $\frac{d}{dt}[t(\tilde{J}_\omega(\phi_t) - \frac{1}{V}\int_M \phi_t e^{-\sigma(u_\omega)}\omega^n] = -(\tilde{I}_\omega - \tilde{J}_\omega)(\phi_t)$. By direct computation,

$$\begin{aligned}
\frac{d}{dt}[t(\tilde{J}_\omega(\phi_t) - \frac{1}{V}\int_M \phi_t e^{-\sigma(u_\omega)}\omega^n)] &= \frac{d}{dt}(t\frac{1}{V}\int_0^t\int_M \dot{\phi}_s e^{-\sigma(u_{\omega_s})}\omega_{\phi_s}^n ds) \\
&= \frac{1}{V}\int_0^t\int_M \dot{\phi}_s e^{-\sigma(u_{\omega_s})}\omega_{\phi_s}^n ds + t\frac{1}{V}\int_M \dot{\phi}_t e^{-\sigma(u_{\omega_t})}\omega_{\phi_t}^n
\end{aligned}$$

$$\begin{aligned}
-(\tilde{I}_\omega - \tilde{J}_\omega)(\phi_t) &= \frac{1}{V}\int_M \phi_t(e^{-\sigma(u_\omega)}\omega^n - e^{-\sigma(u_{\omega_t})}\omega_{\phi_t}^n) - \frac{1}{V}\int_0^t\int_M \dot{\phi}_s(e^{-\sigma(u_\omega)}\omega^n - e^{-\sigma(u_{\omega_s})}\omega_{\phi_s}^n)ds \\
&= \frac{1}{V}\int_0^t\int_M \dot{\phi}_s e^{-\sigma(u_{\omega_s})}\omega_{\phi_s}^n ds - \frac{1}{V}\int_M \phi_t e^{-\sigma(u_{\omega_t})}\omega_{\phi_t}^n
\end{aligned}$$

Differentiating equation (2.4) with respect to t and using the relation (2.8) one gets $-\tilde{\Box}_{\omega\phi_t}\dot{\phi} = \phi + t\dot{\phi}$. By (2.6) we derive that $0 = -\int_M(\tilde{\Box}_{\omega\phi_t}\dot{\phi})e^{-\sigma(u_{\omega_t})}\omega_{\phi_t}^n = \int_M(\phi + t\dot{\phi})e^{-\sigma(u_{\omega_t})}\omega_{\phi_t}^n$. Hence $-\frac{1}{V}\int_M \phi_t e^{-\sigma(u_{\omega_t})}\omega_{\phi_t}^n = t\frac{1}{V}\int_M \dot{\phi}_t e^{-\sigma(u_{\omega_t})}\omega_{\phi_t}^n$ and the desired equation holds.

Now we are in position to prove Theorem 1.

Proof of Theorem 1: By Proposition 1, $\tilde{F}_\omega(\phi_t) = -\frac{1}{t}\int_0^t(\tilde{I}_\omega - \tilde{J}_\omega)(\phi_s))ds - \log\big(\frac{1}{V}\int_M e^{h_\omega-\phi_t}\omega^n\big)$. Also from fact 1 the first term is negative, hence for $t > \varepsilon$ where ε is a fixed positive constant,

$$\begin{aligned}
\tilde{F}_\omega(\phi_t) &\leq -\log\big(\frac{1}{V}\int_M e^{h_\omega-\phi_t}\omega^n\big) \\
&= -\log\big(\frac{1}{V}\int_M e^{(t-1)\phi_t}\omega_{\phi_t}^n\big) \\
&\leq \frac{1-t}{V}\int_M \phi_t\omega_{\phi_t}^n \\
&= \frac{1-t}{V}\int_M \phi_t e^{-t\phi_t + h_\omega + \sigma(u_{\omega_t})}\omega^n \\
&\leq \frac{1-t}{V}\int_{\{\phi_t>0\}} \phi_t e^{-t\phi_t + h_\omega + \sigma(u_{\omega_t})}\omega^n \\
&\leq C
\end{aligned}$$

where we used the concavity of $\log$ and the fact that xe^{-tx} is uniformly bounded for $x > 0$. Here the constant C only depends on the choice of ε and the initial metric. Hence by the properness of the $\tilde{F}_\omega(\phi_t)$ the functional $\tilde{J}_\omega(\phi_t)$ is uniformly bounded for $t > \varepsilon$. Consequently we have the bound for $osc(\phi_t)$. Next, consider the equation (2.4), which implies that

$$\int_M e^{-\sigma(u_{\omega_t})}\omega_{\phi_t}^n = V = \int_M \omega_{\phi_t}^n = \int_M e^{h_\omega - t\phi_t + \sigma(u_{\omega_t})}\omega^n$$

By mean value theorem, there exists a point x_t on M such that $h_\omega(x_t) - t\phi_t(x_t) + \sigma(u_{\omega_t}(x_t)) = 0$ for each time t. Hence, we get

$$\begin{aligned}
|h_\omega(x) - t\phi_t(x) + \sigma(u_{\omega_t})| &= |(h_\omega(x) - t\phi_t(x) + \sigma(u_{\omega_t}(x))) - (h_\omega(x_t) - t\phi_t(x_t) + \sigma(u_{\omega_t}(x_t)))| \\
&\leq t osc(\phi_t) + 2|h_\omega|_{C^0} + 2|\sigma|_{C^0} \\
&\leq C
\end{aligned}$$

So for $t > \varepsilon$, one gets that ϕ_t is uniformly bounded. The desired C^0 estimate is established.

3 Proof of Theorem 2

Assume that there exists an Einstein-Mabuchi metric ω_{EM} on M, In [9], Mabuchi has proved that

a). The Einstein-Mabuchi metric is unique modulo the action of $Z^0(X)$.

b). There exists a one-parameter family of solutions of (2.4) $\phi_t \in \mathcal{H}_X$, $0 \leq t \leq 1$ such that $\omega_{\phi_1} = \omega_{EM}$.

Now fix a K-invariant potential $\phi \in \mathcal{H}_X$ and set $\omega = \omega_{EM} + \sqrt{-1}\partial\bar{\partial}\phi$. Consider the complex Monge-Ampère equation:

$$\omega_{\phi_t}^n = e^{h_\omega + \sigma(u_{\omega_{\phi_t}}) - t\phi_t}\omega^n \tag{3.1}$$

By above result, we have K-invariant solution for all $t \in [0, 1]$ and $\omega_{\phi_1} = \omega_{EM}$. In particular ϕ_1 and $-\phi$ differ by a constant.

From the section 2 we have seen that the generalized functionals have similar properties as the classical ones. So by similar computation as in [10], we can derive similar inequalities for $\tilde{F}$, and $\tilde{J}$ which are

$$|\tilde{J}_\omega(\phi_1) - \tilde{J}_\omega(\phi_0)| \leq 2osc(\phi_1 - \phi_0) \qquad for\ \phi_0,\ \phi_1 \in \mathcal{H}_X \tag{3.2}$$

$$\tilde{F}_{\omega_{EM}}(\phi) \geq c_0(1-t)\tilde{J}_{\omega_{EM}}(\phi) - c_1(1-t)osc(\phi_t - \phi_1) \tag{3.3}$$

where $c_0,\ c_1$ are constant only depending on the choice of σ.

Then we need to estimate $||\phi_t-\phi_1||_{C^0}$. Rewrite the equation by using ω_{EM} as the reference metric,

$$\log\frac{\omega_{EM}^n}{[\omega_{EM}-\sqrt{-1}\partial\bar{\partial}(\phi_1-\phi_t)]^n}+(\phi_1-\phi_t)+\sigma(u_{\omega_{\phi_t}})-\sigma(u_{\omega_{\phi_1}})=(t-1)\phi_t \tag{3.4}$$

by using the relation $u_{\omega_{\phi_t}}=u_{\omega_{\phi_1}}-\sqrt{-1}\bar{X}(\phi_t-\phi_1)$, we have

$$\log\frac{\omega_{EM}^n}{[\omega_{EM}-\sqrt{-1}\partial\bar{\partial}(\phi_1-\phi_t)]^n}+(\phi_1-\phi_t)+\sigma(u_{\omega_{\phi_1}}+\sqrt{-1}\bar{X}(\phi_1-\phi_t))-\sigma(u_{\omega_{\phi_1}})=(t-1)\phi_t \tag{3.5}$$

The linearization of the left hand side of equation at $\psi=\phi_1-\phi_t=0$ is $\delta\psi\to\Box_{\omega_{EM}}\delta\psi+\delta\psi$. Consider the following space,

$$\Lambda_1(M,\omega_{EM})=\{u\in C^\infty(M)|\Box_{\omega_{EM}}u=\Delta_{\omega_{EM}}u+\sqrt{-1}\dot{\sigma}(u_{\omega_{EM}})\bar{X}u=-u\}$$

Similar to the classical case, one can prove that $\Lambda_1(M,\omega_{EM})$ is isomorphic to a subspace of all homomorphic vector fields on M. Assume that $u_1,\ u_2,\ ...u_m$ form a basis of this space. Define matrix $k_{ij}(g)$ for each $g\in K$ by $\rho(g)u_i=k_{ij}(g)u_j$, where ρ is the action of K on space $\Lambda_1(M,\omega_{EM})$. Under our assumption that ϕ is K-invariant, we have

$$V_i=\int_M(\phi u_i)e^{-\sigma(u_{\omega_{EM}})}\omega_{EM}^n=k_{ij}(g)\int_M\phi u_je^{-\sigma(u_{\omega_{EM}})}\omega_{EM}^n=k_{ij}(g)V_j$$

This implies that the vector $V=(V_1,\ V_2,...,\ V_m)$ is fixed by K. Since K has finite centralizer in G whose Lie algebra is the set of all holomorphic vector fields on M, the vector V must be 0. This is equivalent to say that all K-invariant functions are perpendicular to the space $\Lambda_1(M,\omega_{EM})$. So the linerized operator is invertible for all K-invariant functions. So we can apply the implicit function theorem to estimate $||\phi_t-\phi_1||_{C^0}$ in terms of $(t-1)\phi_t$. Following [10], we need to prove the the following which is similar to lemma 1 in [10], i.e.,

$$||\phi_t-\phi_1||_{C^0}\le C[(1-t)||\phi_t||_{C^0}+1] \tag{3.6}$$

for all $t\in[t_0,1]$, where t_0 (depending on ϕ) is defined by

$$(1-t_0)^{1-\alpha}(1+2(1-t_0)||\phi_{t_0}||_{C^0})^\alpha=\sup_{t\in[t_0,1]}(1-t)^{1-\alpha}(1+2(1-t)||\phi_t||_{C^0})^\alpha=D. \tag{3.7}$$

and D is a constant only depending on the choice of p and κ. Here $p>2n$, $0<\kappa<1$ and $\alpha=\frac{p+\kappa-2}{p-1}$.

Remark: With this bound and Theorem [M1] in the previous section, one can prove Theorem 2 by using the same argument in [10] since the properties of the functionals are similar to the classical case. Since we have an additional term σ in this case, we need to estimate $h_{\omega_t}+\sigma(u_{\omega_t})$ and use the volume form $e^{-\sigma(u_{\omega_t})}\omega_t^n$ instead of h_{ω_t} and ω_t^n. The most important tool to derive the above bound in [10] is the Kähler-Ricci flow. We will

introduce a heat flow and derive the same smoothing lemma for $h_{\omega_t} + \sigma(u_{\omega_t})$ in this case. The bound for $||\phi_t - \phi_1||_{C^0}$ is an easy consequence which can be proved by the method in [10].

For each t consider the following heat flow $f_{s,t}$ in time s with initial data $f_{0,t} = 0$,

$$\frac{\partial f_{s,t}}{\partial s} = \log \frac{(\omega_{\phi_t} + \sqrt{-1}\partial\bar{\partial} f_{s,t})^n}{\omega^n_{\phi_t}} - h_{\omega_{\phi_t}} + f_{s,t} - \sigma(u_{\omega_{\phi_t + f_{s,t}}}) \tag{3.8}$$

which is the same as

$$\frac{\partial \omega_{\phi_t + f_{s,t}}}{\partial s} = -Ric(\omega_{\phi_t + f_{s,t}}) + \omega_{\phi_t + f_{s,t}} - \sqrt{-1}\partial\bar{\partial}\sigma(u_{\omega_{\phi_t + f_{s,t}}}) \tag{3.9}$$

Write f_t for $f_{1,t}$, and consider the Kähler form

$$\omega_{\phi_t + f_t} = \omega + \sqrt{-1}\partial\bar{\partial}(\phi_t + f_t) = \omega_{EM} - \sqrt{-1}\partial\bar{\partial}(\phi_1 - \phi_t - f_t) \tag{3.10}$$

There exists a constant so that

$$\log \frac{\omega^n_{EM}}{[\omega_{EM} - \sqrt{-1}\partial\bar{\partial}(\phi_1 - \phi_t - f_t)]^n} + (\phi_1 - \phi_t - f_t - a_t) + \sigma(u_{\omega_{\phi_t + f_t}}) - \sigma(u_{\omega_{\phi_1}}) = h_{\omega_{\phi_t + f_t}} + \sigma(u_{\omega_{\phi_t + f_t}}) \tag{3.11}$$

which can be seen easily by applying $\sqrt{-1}\partial\bar{\partial}$ to both sides.

Notice that this heat flow contains an additional term involving function σ which may cause trouble for computation. But as long as σ is convex, we can still handle it in the computation. For convenience, let $\eta_0 = \omega_{\phi_t}$, $\eta_s = \eta_0 + \sqrt{-1}\partial\bar{\partial} f$, $h_s + \sigma(u_s) = h_{\eta_s} + \sigma(u_{\eta_s})$. Then $h_s + \sigma(u_s) = -\dot{f} + c_s$ for some constant c_s with $c_0 = 0$. We will use s to indicate norms that are defined with respect to the metric η_s. Then we prove

Lemma 1 *We have the following inequalities*:

$$\begin{array}{ll} (a) & ||\dot{f}||_{C^0} \leq e^s ||h_0 + \sigma(u_0)||_{C^0} \\ (b) & \sup_M (|\dot{f}|^2 + s|\nabla \dot{f}|^2_s) \leq e^{2s} ||h_0 + \sigma(u_0)||^2_{C^0} \\ (c) & \Box_s(h_s + \sigma(u_s)) \geq e^s \Box_0(h_0 + \sigma(u_0)) \end{array}$$

Proof of Lemma 1.Differentiating the flow one get

$$\frac{\partial}{\partial s}\dot{f} = \Box_s \dot{f} + \dot{f}, \tag{3.12}$$

hence $||\dot{f}||_{C^0} \leq e^s ||h_0 + \sigma(u_0)||_{C^0}$, giving (a).

Similarly, we compute the flow for $|\nabla \dot f|_s^2$

$$\begin{aligned}\frac{\partial}{\partial s}|\nabla \dot f|_s^2 &= \frac{\partial}{\partial s}(g^{i\bar j}\dot f_i \dot f_{\bar j}) \\ &= -g^{i\bar a}(\frac{\partial}{\partial s}g_{\bar a}b)g^{b\bar j}\dot f_i \dot f_{\bar j} + g^{i\bar j}\ddot f_i \dot f_{\bar j} + g^{i\bar j}\dot f_i \ddot f_{\bar j}\end{aligned}$$

Use the flow we get that $\frac{\partial}{\partial s}g_{\bar a b} = -R_{\bar a b} + g_{\bar a b} - \partial_b\partial_{\bar a}\sigma(u_{\omega_f})$, so the first term in the above equation becomes

$$\begin{aligned}-g^{i\bar a}(\frac{\partial}{\partial s}g_{\bar a b})g^{b\bar j}\dot f_i \dot f_{\bar j} &= -g^{i\bar a}(-R_{\bar a b} + g_{\bar a b} - \partial_{\bar a}\partial_b\sigma(u_{\omega_f}))g^{b\bar j}\dot f_i \dot f_{\bar j} \\ &= -|\nabla \dot f|_s^2 + g^{b\bar j}R_b^i \dot f_i \dot f_{\bar j} - \sqrt{-1}g^{i\bar a}(\sqrt{-1}\dot\sigma g^{b\bar j}\partial_b u_{\omega_f})_{\bar a}\dot f_i \dot f_{\bar j} \\ &= -|\nabla \dot f|_s^2 + g^{b\bar j}R_b^i \dot f_i \dot f_{\bar j} - \sqrt{-1}g^{i\bar a}(\dot\sigma \overline{X^j})_{\bar a}\dot f_i \dot f_{\bar j} \\ &= -|\nabla \dot f|_s^2 + g^{b\bar j}R_b^i \dot f_i \dot f_{\bar j} - \sqrt{-1}\ddot\sigma g^{i\bar a}\partial_{\bar a}u_{\omega_f}\dot f_i(\bar X \dot f) - \sqrt{-1}\dot\sigma g^{i\bar a}\overline{X^j}_{\bar a}\dot f_i \dot f_{\bar j} \\ &= -|\nabla \dot f|_s^2 + g^{b\bar j}R_b^i \dot f_i \dot f_{\bar j} + \ddot\sigma(X\dot f)(\bar X \dot f) - \sqrt{-1}\dot\sigma g^{i\bar a}\overline{X^j}_{\bar a}\dot f_i \dot f_{\bar j}\end{aligned}$$

Since $\frac{\partial}{\partial s}\dot f = (\Delta_s + \sqrt{-1}\dot\sigma \bar X)\dot f + \dot f$ we have

$$\ddot f_i = g^{l\bar k}\dot f_{\bar k l i} + \dot f_i + \sqrt{-1}\ddot\sigma \partial_i u_{\omega_f}(\bar X \dot f) + \sqrt{-1}\dot\sigma(\bar X \dot f)_i$$

Then

$$\begin{aligned}g^{i\bar j}\ddot f_i \dot f_{\bar j} &= g^{i\bar j}g^{l\bar k}\dot f_{\bar k l i}\dot f_{\bar j} + |\nabla \dot f|_s^2 + \sqrt{-1}\ddot\sigma g^{i\bar j}\partial_i u_{\omega_f}(\bar X \dot f)\dot f_{\bar j} + \sqrt{-1}\dot\sigma g^{i\bar j}(\bar X \dot f)_i \dot f_{\bar j} \\ &= g^{i\bar j}g^{l\bar k}\dot f_{l\bar k i}\dot f_{\bar j} + |\nabla \dot f|_s^2 + \ddot\sigma(\bar X \dot f)^2 + \sqrt{-1}\dot\sigma g^{i\bar j}\overline{X^\alpha}\dot f_{\bar\alpha i}\dot f_{\bar j}\end{aligned}$$

where we use the fact that X is a holomorphic vector field and the relation (1.1) in the last line. Similarly

$$\begin{aligned}\ddot f_{\bar j} &= g^{l\bar k}\dot f_{\bar k l \bar j} + \dot f_{\bar j} + \sqrt{-1}\ddot\sigma \partial_{\bar j}u_{\omega_f}(\bar X \dot f) + \sqrt{-1}\dot\sigma(\bar X \dot f)_{\bar j} \\ &= g^{l\bar k}\dot f_{\bar j \bar k l} - R_{\bar j}^{\bar m}\dot f_{\bar m} + \dot f_{\bar j} + \sqrt{-1}\ddot\sigma \partial_{\bar j}u_{\omega_f}(\bar X \dot f) + \sqrt{-1}\dot\sigma(\bar X \dot f)_{\bar j} \\ g^{i\bar j}\dot f_i \ddot f_{\bar j} &= g^{i\bar j}g^{l\bar k}\dot f_i \dot f_{\bar j \bar k l} - g^{i\bar j}R_{\bar j}^{\bar m}\dot f_{\bar m}\dot f_i + |\nabla \dot f|_s^2 - \ddot\sigma(\bar X \dot f)(X\dot f) \\ &\quad + \sqrt{-1}\dot\sigma g^{i\bar j}\overline{X^\alpha}_{\bar j}\dot f_{\bar\alpha}\dot f_i + \sqrt{-1}\dot\sigma g^{i\bar j}\overline{X^\alpha}\dot f_{\bar\alpha \bar j}\dot f_i\end{aligned}$$

Combing these terms we get that

$$\begin{aligned}\frac{\partial}{\partial s}|\nabla \dot f|_s^2 &= g^{i\bar j}g^{l\bar k}\dot f_{i\bar k l}\dot f_{\bar j} + g^{i\bar j}g^{l\bar k}\dot f_i \dot f_{\bar j \bar k l} + |\nabla \dot f|_s^2 + \ddot\sigma(\bar X \dot f)^2 \\ &\quad + \sqrt{-1}\dot\sigma g^{i\bar j}\overline{X^a}\dot f_{\bar j \bar a}\dot f_i + \sqrt{-1}\dot\sigma g^{i\bar j}\overline{X^a}\dot f_{i\bar a}\dot f_{\bar j}\end{aligned}$$

Also we have

$$\begin{aligned}\square_s|\nabla \dot f|_s^2 &= (\Delta_s + \sqrt{-1}\dot\sigma \bar X)(g^{i\bar j}\dot f_i \dot f_{\bar j}) \\ &= g^{i\bar j}g^{l\bar k}\dot f_{i\bar k l}\dot f_{\bar j} + g^{i\bar j}g^{l\bar k}\dot f_i \dot f_{\bar j \bar k l} + |\nabla\nabla \dot f|^2 + |\nabla\bar\nabla \dot f|^2 \\ &\quad + \sqrt{-1}\dot\sigma g^{i\bar j}\overline{X^a}\dot f_{i\bar a}\dot f_{\bar j} + \sqrt{-1}\dot\sigma g^{i\bar j}\overline{X^a}\dot f_i \dot f_{\bar j \bar a}\end{aligned}$$

Then the flow for $|\nabla\dot{f}|_s^2$ is

$$\frac{\partial}{\partial s}|\nabla\dot{f}|_s^2 = \Box_s|\nabla\dot{f}|_s^2 - |\nabla\nabla\dot{f}|_s^2 - |\nabla\bar{\nabla}\dot{f}|_s^2 + |\nabla\dot{f}|_s^2 + \ddot{\sigma}(\bar{X}\dot{f})^2 \tag{3.13}$$

Since along the flow f is also invariant under the X_R, then $(\bar{X}\dot{f})^2 = -(X_I\dot{f})^2$ where X_I is the imaginary part of holomorphic vector field X. Also by the convexity of σ the last term of the above equation is less than 0, so

$$\frac{\partial}{\partial s}|\nabla\dot{f}|_s^2 \leq \Box_s|\nabla\dot{f}|_s^2 - |\nabla\nabla\dot{f}|_s^2 - |\nabla\bar{\nabla}\dot{f}|_s^2 + |\nabla\dot{f}|_s^2 \tag{3.14}$$

Next we compute the flow for $\dot{f}^2$.

$$\begin{aligned}\frac{\partial}{\partial s}\dot{f}^2 &= 2\ddot{f}\dot{f} \\ &= 2\Box_s\dot{f}\dot{f} + 2\dot{f}^2\end{aligned}$$

$$\begin{aligned}\Box_s\dot{f}^2 &= (\Delta_s + \sqrt{-1}\dot{\sigma}\bar{X})\dot{f}^2 \\ &= 2(\Delta_s + \sqrt{-1}\dot{\sigma}\bar{X})\dot{f}\dot{f} + |\nabla\dot{f}|_s^2\end{aligned}$$

so the flow for $\dot{f}^2$ is

$$\frac{\partial}{\partial s}\dot{f}^2 = \Box_s\dot{f}^2 - 2|\nabla\dot{f}|_s^2 + 2\dot{f}^2 \tag{3.15}$$

Combing these two flows,

$$\frac{\partial}{\partial s}(|\dot{f}|^2 + s|\nabla\dot{f}|_s^2) \leq \Box_s(|\dot{f}|^2 + s|\nabla\dot{f}|_s^2) + 2(|\dot{f}|^2 + s|\nabla\dot{f}|_s^2) \tag{3.16}$$

The maximum principle implies

$$\sup_M(|\dot{f}|^2 + s|\nabla\dot{f}|_s^2) \leq e^{2s}||h_0 + \sigma(u_0)||_{C^0}^2 \tag{3.17}$$

which proves (b).

For $\Box_s\dot{f}$,

$$\begin{aligned}\frac{\partial}{\partial s}(\Box_s\dot{f}) &= \frac{\partial}{\partial s}(\Delta_s + \sqrt{-1}\dot{\sigma}\bar{X})\dot{f} \\ &= (\Box_s(\Delta_s + \sqrt{-1}\dot{\sigma}\bar{X})\ddot{f}_s - g^{i\bar{a}}(\frac{\partial}{\partial s}g_{\bar{a}b})g^{b\bar{j}}\dot{f}_{\bar{j}i} - \sqrt{-1}\ddot{\sigma}(X\dot{f})(\bar{X}\dot{f}) \\ &= \Box_s^2\dot{f} + \Box_s\dot{f} - \ddot{\sigma}(X\dot{f})(\bar{X}\dot{f}) - g^{i\bar{a}}g^{b\bar{j}}\dot{f}_{\bar{j}i}\dot{f}_{\bar{a}b} \\ &= \Box_s^2\dot{f} + \Box_s\dot{f} - \ddot{\sigma}(X\dot{f})(\bar{X}\dot{f}) - |\nabla\bar{\nabla}\dot{f}|_s^2\end{aligned}$$

where we use the flow for $\dot{f}$ and the fact that $\frac{\partial}{\partial s} g_{\bar{a}b} = \dot{f}_{\bar{a}b}$. Hence we get

$$\frac{\partial}{\partial s}\Box_s\dot{f} = \Box_s^2\dot{f} + \Box_s\dot{f} - \ddot{\sigma}(X\dot{f})(\bar{X}\dot{f}) - |\nabla\bar{\nabla}\dot{f}|_s^2 \le \Box_s^2\dot{f} + \Box_s\dot{f} \tag{3.18}$$

and c) also follows from the maximum principle.

Lemma 2 *Let* $v = (h_1 + \sigma(u_1)) - \frac{1}{V}\int_M (h_1 + \sigma(u_1))e^{-\sigma(u_1)}\eta_1^n$*, then for any* $p > 2n$*, there exists constant* $C > 0$*, depending only on* ω_{EM}*,* σ *and* p *so that*

$$||v||_{C^0} \le C||h_0 + \sigma(u_0)||_{C^0}^{\frac{p-2}{p-1}}(1-t)^{\frac{1}{p-1}} \tag{3.19}$$

Proof of Lemma 2. Lemma 1 shows that

$$||v||_{C^0} \le 2e||h_0 + \sigma(u_0)||_{C^0} \tag{3.20}$$

Since v is a real-valued function and X_R-invariant, we also have

$$\begin{aligned}
\int_M |\nabla v|_1^2 e^{-\sigma(u_1)}\eta_1^n &= -\int_M v(\Box_1 v)e^{-\sigma(u_1)}\eta_1^n \\
&= \int_M (v - \inf_M v)(-\Box_1 v)e^{-\sigma(u_1)}\eta_1^n \\
&\le \int_M (v - \inf_M v)\sup_M(-\Box_1 v)e^{-\sigma(u_1)}\eta_1^n \\
&\le 2V||v||_{C^0}\sup_M(-\Box_1 v)
\end{aligned}$$

Recall that $h_0 + \sigma(u_0) = h_{\omega_{\phi_t}} + \sigma(u_{\omega_{\phi_t}})$ and thus $Ric(\eta_0) + \sqrt{-1}\partial\bar{\partial}\sigma(u_{\eta_1}) > t\eta_0$ which implies that $\Delta_0(h_0 + \sigma(u_0)) \ge -n(1-t)$. Also one has $h_{\omega_{\phi_t}} + \sigma(u_{\omega_{\phi_t}}) = -(1-t)\phi_t + c_t$ then

$$\begin{aligned}
|\sqrt{-1}\dot{\sigma}\bar{X}(h_{\omega_{\phi_t}} + \sigma(u_{\omega_{\phi_t}}))| &\le C(1-t)|X\phi_t| \\
&= C(1-t)|u_{\omega_{\phi_t}} - u_\omega| \\
&\le C(1-t)
\end{aligned}$$

where we used the fact that $max_M u$ and $min_M u$ are holomorphic invariant and constant C depends on σ. Hence $\Box_0 h_0 + \sigma(u_0) = (\Delta_0 + \sqrt{-1}\dot{\sigma}\bar{X})(h_0 + \sigma(u_0)) \ge -(C+n)(1-t)$. So by Lemma 1,

$$-\Box_1(h_1 + \sigma(u_1)) \le -(C+n)e(1-t) \tag{3.21}$$

Substituting in the previous inequality gives

$$\int_M |\nabla v|_1^2 e^{-\sigma(u_1)}\eta_1^n \le 2V_0(C+n)e||v||_{C^0}(1-t) \tag{3.22}$$

Let $p > 2n$. Then some constant C_i depending only on ω_{EM}, σ, A and p,

$$\begin{aligned} ||v||^p_{C^0} &\leq C(\int_M |v|^p e^{-\sigma(u_1)}\eta_1^n + \int_M |\nabla v|_1^p e^{-\sigma(u_1)}\eta_1^n) \\ &\leq C_0(||v||^{p-2}_{C^0}\int_M |v|^2 e^{-\sigma(u_1)}\eta_1^n + (e||h_0+\sigma(u_0)||_{C^0})^{p-2}\int_M |\nabla v|_1^2 e^{-\sigma(u_1)}\eta_1^n) \\ &\leq C_1||h_0+\sigma(u_0)||^{p-2}_{C^0}\int_M |\nabla v|_1^2 e^{-\sigma(u_1)}\eta_1^n \end{aligned}$$

where we have used the Sobolev inequality, the Poincaré inequality and applied (b) of Lemma 1. Here the constants in the Sobolev and Poincaré inequalities depend only on ω_{EM} since the metric η_1 is equivalent to ω_{EM}. Together with inequality (3.22), this gives

$$||v||^p_{C^0} \leq C_2(1-t)||h_0+\sigma(u_0)||^{p-2}_{C^0}||v||_{C^0} \tag{3.23}$$

which is the inequality to be proved.

With the help of the above two smoothing lemmas we can use the same argument as in [10] to derive the bound for $||\phi_t - \phi_1||_{C^0}$, then the Theorem 2 follows at once by our remark at the beginning of the section.

4 A holomorphic invariant of Futaki Type

As an analogue of the Futaki invariant, we can also define an invariant of this type which can be seen as an obstruction to the existence of Einstein-Mabuchi metrics.

Let $\eta(M)$ be the complex Lie algebra which consists of all holomorphic vector fields on M. Then we define the functional associated to multiplier Hermitian structure below,

$$F^\sigma_X(V) = \int_M V(h_\omega + \sigma(u_\omega))e^{-\sigma(u_\omega)}\omega^n,\ V \in \eta(M).\ \ \textit{and } \omega \in \mathcal{K}_X \tag{4.1}$$

If there exists an Einstein-Mabuchi metric on M, the above functional vanishes. When $X = 0$, the above functional coincides with the Futaki invariant. The following Theorem shows that the functional is well-defined and it is a holomorphic invariant on M.

Theorem 3 *The functional F^σ_X is independent of the choice of ω with $\omega \in \mathcal{K}_X$.*
Proof. Let ω' be another Kähler form in $C_1(M)$. Assume that $\omega_s = \omega + \sqrt{-1}\partial\bar{\partial}\phi(s)$ where $\phi(s) \in \mathcal{H}_X$ for $0 \leq s \leq 1$ is a path connecting ω and ω' with $\phi(0) = 0$ and $\omega_1 = \omega'$. Along the path, we have

$$\frac{d}{ds}(h_s + \sigma(u_s)) = -\dot{\phi} - \Box_s\dot{\phi} \tag{4.2}$$

To the homomorphic vector field V one can associate a smooth complex-valued function v such that $V^\alpha = g^{\alpha\bar{\beta}}\partial_{\bar{\beta}}v$. Note here we don't require v to be real-valued as in (1.1). For

a real-valued function f, one has that $V(f) = g^{\alpha\bar{\beta}}\partial_{\bar{\beta}}v\partial_\alpha f = (\bar{\partial}v, \bar{\partial}f)_\omega$. By (2.6) and $h_s + \sigma(u_s)$ is real-valued, we compute the derivative of F^σ_X,

$$\begin{aligned}
\frac{d}{ds}F^\sigma_X(V) &= \int_M V(\frac{d}{ds}(h_s+\sigma(u_s)))e^{-\sigma(u_{\omega_s})}\omega_s^n + \int_M V(h_s+\sigma(u_s))\square_s\dot{\phi}e^{-\sigma(u_{\omega_s})}\omega_s^n \\
&= \int_M (\bar{\partial}v, \bar{\partial}(-\dot{\phi}-\square_s\dot{\phi}))_{\omega_s}e^{-\sigma(u_{\omega_s})}\omega_s^n + \int_M V(h_s+\sigma(u_s))\square_s\dot{\phi}e^{-\sigma(u_{\omega_s})}\omega_s^n \\
&= \int_M \overline{(\bar{\partial}(-\dot{\phi}-\square_s\dot{\phi}), \bar{\partial}v)}e^{-\sigma(u_{\omega_s})}\omega_s^n + \int_M V(h_s+\sigma(u_s))\square_s\dot{\phi}e^{-\sigma(u_{\omega_s})}\omega_s^n \\
&= \int_M (\dot{\phi}+\square_s\dot{\phi})\square_s v e^{-\sigma(u_{\omega_s})}\omega_s^n + \int_M V(h_s+\sigma(u_s))\square_s\dot{\phi}e^{-\sigma(u_{\omega_s})}\omega_s^n \\
&= \int_M \dot{\phi}\square_s v e^{-\sigma(u_{\omega_s})}\omega_s^n + \int_M \square_s\dot{\phi}\{\square_s v + V(h_s+\sigma(u_s))\}e^{-\sigma(u_{\omega_s})}\omega_s^n
\end{aligned}$$

We need to use integration by part for the first term, notice that ϕ and $\square_s\phi$ are real-valued, then

$$\begin{aligned}
\int_M \dot{\phi}\square_s v e^{-\sigma(u_{\omega_s})}\omega_s^n &= \overline{\int_M \dot{\phi}\overline{\square_s v}e^{-\sigma(u_{\omega_s})}\omega_s^n} \\
&= -\int_M \overline{(\bar{\partial}\dot{\phi}, \bar{\partial}v)}_{\omega_s}e^{-\sigma(u_{\omega_s})}\omega_s^n \\
&= -\int_M (\bar{\partial}v, \bar{\partial}\dot{\phi})_{\omega_s}e^{-\sigma(u_{\omega_s})}\omega_s^n \\
&= \int_M v\overline{\square_s\dot{\phi}}e^{-\sigma(u_{\omega_s})}\omega_s^n \\
&= \int_M v\square_s\dot{\phi}e^{-\sigma(u_{\omega_s})}\omega_s^n
\end{aligned}$$

hence

$$\begin{aligned}
\frac{d}{ds}F^\sigma_X(V) &= \int_M \square_s\dot{\phi}v e^{-\sigma(u_{\omega_s})}\omega_s^n + \int_M \square_s\dot{\phi}\{\square_s v + V(h_s+\sigma(u_s))\}e^{-\sigma(u_{\omega_s})}\omega_s^n \\
&= \int_M \square_s\dot{\phi}\{v + \square_s v + V(h_s+\sigma(u_s))\}e^{-\sigma(u_{\omega_s})}\omega_s^n \\
&= -\int_M (\bar{\partial}q, \bar{\partial}\dot{\phi})e^{-\sigma(u_{\omega_s})}\omega_s^n
\end{aligned}$$

where $q = v + \square_s v + V(h_s + \sigma(u_s))$. Now we only need to prove that q is holomorphic. First notice that

$$\begin{aligned}
V(h_s+\sigma(u_s)) &= V(h_s) + V(\sigma(u_s)) \\
&= V(h_s) + g^{\alpha\bar{\beta}}\partial_{\bar{\beta}}v\partial_\alpha(\sigma(u_s))
\end{aligned}$$

So we can simplify q as

$$\begin{aligned}
q &= v + \square_s v + V(h_s) + g^{\alpha\bar{\beta}}\partial_{\bar{\beta}}v\partial_\alpha(\sigma(u_s)) \\
&= v + \Delta_s v - g^{\alpha\bar{\beta}}\partial_{\bar{\beta}}v\partial_\alpha(\sigma(u_s)) + V(h_s) + g^{\alpha\bar{\beta}}\partial_{\bar{\beta}}v\partial_\alpha(\sigma(u_s)) \\
&= v + \Delta_s v + V(h_s) \\
&= v + g^{i\bar{j}}v_{\bar{j}i} + g^{i\bar{j}}v_{\bar{j}}h_i
\end{aligned}$$

where we use the definition for $\Box$ in section 2. Then

$$\begin{aligned} q_{\bar{l}} &= v_{\bar{l}} + g^{i\bar{j}} v_{\bar{j}i\bar{l}} + g^{i\bar{j}} v_{\bar{j}\bar{l}} h_i + g^{i\bar{j}} v_{\bar{j}} h_{\bar{l}i} \\ &= v_{\bar{l}} + g^{i\bar{j}} v_{\bar{j}\bar{l}i} - g^{i\bar{j}} v_{\bar{j}} R_{\bar{l}i} + g^{i\bar{j}} v_{\bar{j}} (R_{\bar{l}i} - g_{\bar{l}i}) \\ &= 0 \end{aligned}$$

where we use the fact that h_s is the Ricci potential and $V = g^{i\bar{j}} v_{\bar{j}}$ is a holomorphic vector field, i.e., $g^{i\bar{j}} v_{\bar{j}\bar{l}} = 0$. Thus we prove that along the path the derivative of $F_X^\sigma(V)$ is 0. The theorem follows as well.

Acknowledgements: The author is grateful to his advisor D.H.Phong for his advice and constant support. He also would like to thank Zuoliang Hou for some helpful discussions.

References

[1] Aubin, Thierry Réduction du cas positif de l'équation de Monge-Ampère sur les variétés kählériennes compactes $\grave{a}$ la démonstration d'une inégalité. J. Funct. Anal. 57 (1984), no. 2, 143–153.

[2] Bando, Shigetoshi; Mabuchi, Toshiki Uniqueness of Einstein Kähler metrics modulo connected group actions. Algebraic geometry, Sendai, 1985, 11-40, Adv. Stud. Pure Math 10, North-Holland, Amsterdam, 1987.

[3] Cao, H; Tian, G; Zhu, X Kähler-Ricci solitons on compace complex manifolds with $C_1(M) > 0$.Geom. Funct. Anal. 15 (2005), no. 3.

[4] Donaldson, S.K., "Scalar curvature and stability of toric varieties", J. Differential Geometry **62** (2002) 289-349.

[5] Futaki, Akito; Mabuchi, Toshiki Bilinear forms and extremal Kähler vector fields associated with Kähler classes. Math. Ann. 301 (1995), no. 2, 199–210

[6] Mabuchi, Toshiki Kähler-Einstein metrics for manifolds with nonvanishing Futaki character. Tohoku Math. J. (2) 53 (2001), no. 2, 171–182

[7] Mabuchi, Toshiki A theorem of Calabi-Matsushima's type. Osaka J. Math. 39 (2002), no. 1, 49–57

[8] Mabuchi, Toshiki Heat kernel estimates and the Green functions on multiplier Hermitian manifolds. Tohoku Math. J. (2) 54 (2002), no. 2, 259–275

[9] Mabuchi, Toshiki Multiplier Hermitian structures on Kähler manifolds. Nagoya Math. J. 170 (2003), 73–115

[10] Phong, D.H. , J. Sturm, J. Song and B.Weinkove The Moser-Trudinger inequality on the Kähler-Einstein manfiolds, arXiv: math.DG/0604076.

[11] Tian, G., Kähler-Einstein metrics with positive scalar curvature, Inventiones Math. 130 (1997) 1-37.

[12] Tian, Gang; Zhu, Xiaohua Uniqueness of Kähler-Ricci solitons. Acta Math. 184 (2000), no. 2, 271–305

[13] Tian, Gang; Zhu, Xiaohua A new holomorphic invariant and uniqueness of Kähler-Ricci solitons. Comment. Math. Helv. 77 (2002), no. 2, 297–325.

[14] Yau, S.T., "On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampere equation I", Comm. Pure Appl. Math. 31 (1978) 339-411.

[15] Yau, S.T., "Open problems in geometry", Proc. Symposia Pure Math **54** (1993) 1-28.